Lecture 3

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1 Linear systems and their solutions

This lecture we're going to speak about the most important and boring part of linear algebra — about general linear systems — we will learn how to solve and analyze them.

Definition 1.1. **Linear system** *is a bunch of linear equations considered together:*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{cases} \tag{1}$$

Here we have again 3 types of letters: x_i's are variables which have to be determined, letter a_{ij} represents the coefficient in the i-th equation before the variable x_j, and letter b_i is the constant term in the i-th equation. Both a_{ij}'s and b_i's are given numbers.

The **solution** *of the system is an n-tuple of numbers such that it is a solution of each of the system's equation.*

To solve *the system means to find a set of all its solutions, i.e. solution set.*

Definition 1.2. *The systems are called* **equivalent** *if they have the same solution sets. In other words, two systems are equivalent if any solution of the first system is a solution for the second system and any solution of the second system is a solution for the first system.*

2 Elementary operations: three ways to get equivalent systems

The following three operations can be applied to any linear system to get an equivalent system. They are called **elementary operations.**

1. **Interchanging of the equations.** If we interchange two equations, i.e. rewrite them in different order we'll get the equivalent system — it's obvious.

   Example 2.1. *Let's consider 2 systems:*

$$\begin{cases} x_1 + 2x_2 & = & 4 \\ 2x_1 - x_2 & = & 3 \end{cases}$$

   and

$$\begin{cases} 2x_1 - x_2 & = & 3 \\ x_1 + 2x_2 & = & 4 \end{cases}$$

   The second system is got from the first by interchanging its equations. It is obvious that they have same solutions — $x_1 = 2$ *and* $x_2 = 1$.

2. **Multiplication.** We can multiply any equation by any number not equal to zero.

   Example 2.2. *Let's consider 2 systems:*

$$\begin{cases} x_1 + 2x_2 & = & 4 \\ 2x_1 - x_2 & = & 3 \end{cases}$$

   and

$$\begin{cases} 2x_1 + 4x_2 & = & 8 \\ 2x_1 - x_2 & = & 3 \end{cases}$$

   The second system is obtained from the first by multiplying the first equation by 2. It is obvious that they have same solutions — $x_1 = 2$ *and* $x_2 = 1$.

3. **Addition.** We can add any equation multiplied by some number to any other equation.

   Example 2.3. *Let's consider 2 systems:*

$$\begin{cases} x_1 + 2x_2 & = & 4 \\ 2x_1 - x_2 & = & 3 \end{cases}$$

   and

$$\begin{cases} x_1 + 2x_2 & = & 4 \\ 4x_1 + 3x_2 & = & 11 \end{cases}$$

The second system is obtained from the first by adding the first equation multiplied by 2 to the second equation. It can be seen that they have same solutions — $x_1 = 2$ *and* $x_2 = 1$*.*

The following theorem is the main result about elementary operations.

Theorem 2.4. *Applying an elementary operation to a system gives an equivalent system.*

Proof. It is obvious that every solution of the first ("old") system is the solution for the second ("new") system.

From the other side, "old" system can be obtained from the "new" one by the same type elementary operation. For the first 2 types it is obvious, and for the 3rd type if we are, say, in the "old" system adding the second equation multiplied by c to the first one, we can go back from the "new" system to the "old" one by adding the second equation multiplied by $-c$ to the first one. So, by applying any elementary operation we get the equivalent system. □

3 Row echelon form of a system

We'll start with two definitions.

Definition 3.1. *The variable* x_i *is called* **leading** *for a linear equation*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

if for all j *which are less then* i*,* $a_j = 0$*, i. e.* a_i *is the first nonzero coefficient in the equation.*

Definition 3.2. *The variable* x_i *is called* **free** *for a system (1) if it is not leading for any of the system's equation.*

Example 3.3. *Consider the following system:*

$$\begin{cases} x_1 + 2x_2 + 3x_3 + x_4 = 4 \\ \phantom{x_1 + 2x_2 + {}} 2x_3 - x_4 = 3 \end{cases}$$

For this system x_1 *and* x_3 *are leading variables, and* x_2 *and* x_4 *are free variables.*

Now we'll consider some special type of linear systems.

Definition 3.4. *The system is said to be in* **(row) echelon form** *if the subscripts of the leading variables in its equations form a strictly increasing sequence, and all zero equations (equations of the form* $0x_1 + \cdots + 0x_n = b$*) are at the bottom of the system.*

Example 3.5. *The system*

$$\begin{cases} x_1 + 2x_2 + 3x_3 + x_4 = 4 \\ 5x_2 + 2x_3 - x_4 = 3 \\ 2x_4 = 10 \end{cases}$$

is in row echelon form, since the sequence of the subscripts of its leading variables is $1, 2, 4$ *— strictly increasing, and the system*

$$\begin{cases} x_1 + 2x_2 + 3x_3 + x_4 = 4 \\ 5x_2 + 2x_3 - x_4 = 3 \\ 3x_2 + 2x_4 = 10 \end{cases}$$

is not in row echelon form since the sequence of the subscripts of its leading variables is $1, 2, 2$ *— not strictly increasing.*

The algorithm of solving linear system consists of the following 2 steps:

Step 1 Reduce the system to the equivalent system in row echelon form by elementary operations.

Step 2 Solve the system in row echelon form.

4 Gaussian elimination

The algorithm for reducing any system to the equivalent system in row echelon form is called **Gaussian elimination.**

Algorithm [Gaussian Elimination]

1. Let x_{j_1} be the first variable with nonzero coefficient in at least one equation.

2. Interchange the equations such that the equation with nonzero coefficient before x_{j_1} be a first equation. It is a type 1 elementary operation.

3. Add (subtract) the first equation of the system multiplied by corresponding numbers to all other system's equations to get zero coefficients before x_{j_1} in all equation but the first one. This is made by type 3 elementary operations.

4. Apply steps (1)-(3) to all system's equations but the first one.

In this algorithm we are using elementary operations only of the first and the third type, but in practice the operations of the second type may be helpful. Since the elementary operations don't change the solution set of the system, instead of the solving the initial system we can solve the system in row echelon form, obtained by this algorithm, and their solutions will be the same.

Example 4.1. *Let's reduce the following system to the row echelon form.*

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ x_1 + 3x_2 + 2x_3 - x_4 = 4 \\ 2x_1 + x_2 - x_3 + 3x_4 = -2 \\ 2x_1 - 2x_3 + 3x_4 = 1 \end{cases} \tag{2}$$

First variable with nonzero coefficient is x_1, and the first equation has a nonzero coefficient before it too. So, we don't have to interchange the equations. Now, we'll subtract the first equation from the second one, then multiply it by 2 and subtract from the third one, and then multiply it by 2 again and subtract it from the fourth one. We'll get the following system (we should not omit the first equation!!!):

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ x_2 + x_3 - x_4 = 2 \\ -3x_2 - 3x_3 + 3x_4 = -6 \\ -4x_2 - 4x_3 + 3x_4 = -3 \end{cases} \tag{3}$$

Now, we'll apply the same steps to equations from the 2nd to the 4th. So, we multiply the second equation by 3 and add it to the third one, and than multiply the second equation by 4 and add it to the fourth one (again, do not omit any equations!!!):

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ x_2 + x_3 - x_4 = 2 \\ 0 = 0 \\ -x_4 = 5 \end{cases} \tag{4}$$

Now, to get a system in row echelon form we have to interchange the third and fourth equations:

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ x_2 + x_3 - x_4 = 2 \\ -x_4 = 5 \\ 0 = 0 \end{cases} \tag{5}$$

5 Solving the system in REF

Next we will describe the algorithm of solving a system in (row) echelon form. First of all, if the system has zero equation of the form $0x_1 + \cdots + 0x_n = b$, where $b \neq 0$, then the system has no solutions. Than, if the system has zero equations of the form $0x_1 + \cdots + 0x_n = 0$ then we can simply omit them. In this case and in the case when the system has no zero equations, we can use the following algorithm to determine the solution.

Algorithm.

1. Assign arbitrary values, called **parameters** to the free variables, and substitute them to the system.

2. Move the terms with free variables to the right hand side of the equation.

3. Express leading variables in terms of free variables.

The second step of this algorithm is called **back substitution**. It is done from the bottom of the system to the top of it — first we get an expression for the leading variable of the last equation, then we move to the equation before the last one, etc. till we get to the leading variable of the first equation.

Example 5.1. *We will continue solving the system from the example (4.1). The row echelon form of the system is (we omit a zero equation here):*

$$\left\{\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & x_3 & & & = & 2 \\ & & x_2 & + & x_3 & - & x_4 & = & 2 \\ & & & & & - & x_4 & = & 5 \end{array}\right. \tag{6}$$

The leading variables of this system are x_1, x_2, and x_4. The free variable is x_3. We will move the terms with free variable to the right hand side. We will obtain the following system:

$$\left\{\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & & & = & 2 & - & x_3 \\ & & x_2 & - & x_4 & = & 2 & - & x_3 \\ & & & - & x_4 & = & 5 & & \end{array}\right. \tag{7}$$

Now, using back substitution we'll get the values for all unknowns. From the last equation: $x_4 = -5$. Substituting it to the second equation we'll get the value for x_2 in terms of x_3:

$$x_2 = 2 - x_3 + x_4 = 2 - x_3 - 5 = -3 - x_3$$

Than from the first equation by substituting expressions for x_4 and x_2 we can get an expression for x_1:

$$x_1 = 2 - x_3 - 2x_2 = 2 - x_3 - 2(-3 - x_3) = 2 - x_3 + 6 + 2x_3 = 8 + x_3.$$

Now, to write the solution correctly we have to use set notation:

$$\{(x_3 + 8; \quad -x_3 - 3; \quad x_3; \quad -5) \, : \, x_3 \in \mathbb{R}\}$$

It means, that for any value of x_3 we can get a numerical solution, say if, for example, $x_3 = 0$, then the corresponding numerical solution is

$$(8, -3, 0, -5)$$